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Universal 23 symmetry

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Abstract. The possible maximal mixing seen in the oscillations of atmospheric neutrinos has led to the postulate of μ–τ symmetry, which interchanges ν_μ and ν_τ. We argue that such a symmetry need not be special to neutrinos but can be extended to all fermions. The assumption that all fermion mass matrices are approximately invariant under the interchange of the second and the third generation fields is shown to be phenomenologically viable and has interesting consequences. In the quark sector, the smallness of V_{ub} and V_{cb} can be consequences of this approximate 2–3 symmetry. The same approximate symmetry can simultaneously lead to a large atmospheric mixing angle and can describe the leptonic mixing quite well. We identify two generic scenarios leading to this. One is based on the conventional type-I seesaw mechanism and the other follows from the type-II seesaw model. The latter requires a quasi-degenerate neutrino spectrum for obtaining large atmospheric neutrino mixing in the presence of an approximate μ–τ symmetry.

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1 Introduction

The vastly different mixing patterns [1–5] of quarks and leptons have been used as an argument in favor of special leptonic symmetries such as μ–τ interchange [6–34], L_e–L_μ–L_τ [35–46] , D_4 [47–49], A_4 [50–64] symmetry, etc. These symmetries lead to large or maximal mixing angles, seen in the leptonic sector. It is interesting to ask if any of these symmetries are purely leptonic or if they can be extended to describe the quark mixing as well. Any such symmetry will have to explain a small mixing for quarks and simultaneously large mixing among leptons. We argue that a generalization of the μ–τ symmetry (to be called 23 symmetry) that interchanges the second and the third generation fermionic fields provides such an example. This 23 symmetry appears to be more natural in the quark sector. In the 23 symmetric limit, the elements V_{ub} and V_{cb} of the CKM matrix V are zero and their small and hierarchical values can arise from its breaking. In the leptonic sector, the exact 23 symmetry leads to a completely wrong prediction, namely a vanishing atmospheric mixing angle. We discover that this can be avoided and an approximate 23 symmetry broken at the few % level can explain leptonic mixing if the neutrino spectrum is quasi-degenerate. Even without degeneracy, the approximate 23 symmetry at the Lagrangian level can lead to correct understanding of the leptonic mixing in case of the type-I seesaw mechanism for large ranges in parameter space, which we identify.

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2 23 symmetry and quark mixing

Let us first elaborate on the well-known [6–29] consequences of the μ–τ symmetry. The light neutrino mass matrix $\mathcal{M}_\nu$ is restricted to the following form in the presence of this symmetry:

$$\mathcal{M}_\nu = \begin{pmatrix} X_\nu & A_\nu & A_\nu \\ A_\nu & B_\nu & C_\nu \\ A_\nu & C_\nu & B_\nu \end{pmatrix} . \qquad (1)$$

This form leads to a maximal atmospheric mixing and zero U_{e3} if it is assumed to be true in the flavor basis. In the same basis, the charged lepton mass matrix is diagonal and consequently it is not invariant under the μ–τ symmetry, which would have implied $m_\mu = m_\tau$. It is possible to imagine a larger symmetry (e.g. D_4 [47–49]), which when broken leads to the above form for $\mathcal{M}_\nu$ in the flavor basis. In this case, the μ–τ symmetry appears to be only an effective neutrino symmetry.

It is important to stress that the μ–τ symmetry by itself does not force equality of the muon and tau masses. To see this, let us simultaneously assume that both the charged lepton mass matrix M_l and $\mathcal{M}_\nu$ are μ–τ symmetric and have the form[1] given in (1). In this case, the muon and tau masses are different, but now the 23 mixing angle for

[1] The 2–3 symmetry does not automatically imply the form given in (1) for M_l, unless it is assumed to be symmetric. This assumption can easily be realized in the context of GUT such as SO(10) which commutes with the 2–3 symmetry.

the charged leptons is also maximal. As a consequence, the neutrino and the charged lepton mixing angles cancel, and one gets a vanishing atmospheric mixing angle. In either case, the μ–τ symmetry does not appear to be an exact symmetry in the leptonic world.

In contrast to leptons, the 23 and the 13 mixing angles are indeed small for quarks. This suggests that a generalized μ–τ symmetry may be a good symmetry for quarks rather than for leptons. Let us then postulate that the quark mass matrices are symmetric and display an approximate 2–3 symmetry. Later on we will discuss situations in which this assumption can be extended to the leptonic masses as well. An approximate 2–3 symmetry dictates the following form for a symmetric fermion mass matrix M_f:

$$M_f = \begin{pmatrix} X_f & A_f(1-\epsilon_{1f}) & A_f(1+\epsilon_{1f}) \\ A_f(1-\epsilon_{1f}) & B_f(1-\epsilon_{2f}) & C_f \\ A_f(1+\epsilon_{1f}) & C_f & B_f(1+\epsilon_{2f}) \end{pmatrix} . \quad (2)$$

The dimensionless parameters $\epsilon_{1f,2f}$ break the 2–3 symmetry and are assumed to be $\ll 1$. These two parameters are sufficient to describe the most general 2–3 breaking [23–29] when the fermion mass matrices are symmetric.

Let us first consider the symmetric limit, assuming all parameters in (2) to be real. All the eigenvalues of M_f are distinct and are given by

$$\begin{aligned} m_{1f} &= \frac{1}{2}\left[B_f+C_f+X_f-\left((B_f+C_f-X_f)^2+8A_f^2\right)^{1/2}\right] , \\ m_{2f} &= \frac{1}{2}\left[B_f+C_f+X_f+\left((B_f+C_f-X_f)^2+8A_f^2\right)^{1/2}\right] , \\ m_{3f} &= B_f-C_f . \end{aligned} \quad (3)$$

We will assume the hierarchy $|m_{1f}|<|m_{2f}|<|m_{3f}|$ and associate the fermionic states accordingly to these eigenvalues. The M_f can be diagonalized by a matrix V_f^0:

$$V_f^0 = R_{23}(\pi/4)R_{12}(\theta_{12f}) . \quad (4)$$

As a result, one gets in the symmetric limit

$$V^0_{\mathrm{CKM}} = V_u^{0\dagger}V_d^0 = R_{12}(\theta_{\mathrm{C}}) , \quad (5)$$

with

$$\theta_{\mathrm{C}} = \theta_{12d}-\theta_{12u} .$$

It follows from (5) that the 2–3 symmetry automatically leads to vanishing V_{cb} and V_{ub}. This remains true even if M_f is complex. The Cabibbo angle and the quark masses are not restricted by this symmetry. The Cabibbo angle can be constrained by imposing an additional discrete symmetry D, defined by

$$f_{1\mathrm{L}} \to \mathrm{i} f_{1\mathrm{L}} ; \quad f_{1\mathrm{R}} \to -\mathrm{i} f_{1\mathrm{R}} . \quad (6)$$

This symmetry forces A_f and X_f in (2) to be zero. The A_f term breaks this symmetry by one and X_f by two units (of i). B_f and C_f are invariant. It is thus natural to assume that D breaking (by some flavon field) can lead to a hierarchy $|B_f,C_f|\gg|A_f|\gg|X_f|$. This hierarchy leads to $A_f\sim\mathcal{O}(\sqrt{m_{1f}m_{2f}})$ and the celebrated relation [65, 66]

$$\theta_{\mathrm{C}} \sim \sqrt{\frac{m_d}{m_s}}-\sqrt{\frac{m_u}{m_c}} . \quad (7)$$

More precisely, one needs

$$|X_f| \ll |\sqrt{2}A_f| \ll |B_f+C_f| \ll |B_f-C_f| , \quad (8)$$

for $f=u,d$ in order to get (7) and the hierarchical masses. It follows that an approximately broken D and an exact 2–3 symmetry leads to (7) and vanishing V_{ub} and V_{cb}. Subsequent breaking of the 2–3 symmetry can then induce the latter quantities.

While both ϵ_{1f} and ϵ_{2f} could be present in a model, we consider here one parameter breaking for all M_f and assume that only ϵ_{2f} is non-zero. It is straightforward to add the effect of ϵ_{1f}. We will also take all parameters to be real.

The non-zero ϵ_{2u} and ϵ_{2d} are sufficient to generate the required values of V_{ub} and V_{cb}. The M_f can be diagonalized in the limit specified in (8) as follows:

$$V_f^{\mathrm{T}} M_f V_f = \mathrm{Diag.}(m_{1f}, m_{2f}, m_{3f}) ,$$

with

$$\begin{aligned} m_{3f} &\approx B_f-C_f\left(1+\frac{1}{2}\theta_{23f}^2\right) , \\ m_{2f} &\approx B_f+C_f\left(1+\frac{1}{2}\theta_{23f}^2\right)+\frac{2A_f^2}{m_{2f}} , \\ m_{1f} &\approx -\frac{2A_f^2}{m_{2f}} , \end{aligned} \quad (9)$$

where $f=u,d$. The mixing matrix is given as

$$V_f = R_{23}(\pi/4)R_{23}(\theta_{23f})R_{13}(\theta_{13f})R_{12}(\theta_{12f}) , \quad (10)$$

with

$$\begin{aligned} \theta_{23f} &\approx \frac{\epsilon_{2f}B_f}{2C_f} \approx -\frac{\epsilon_{2f}}{2} , \\ \theta_{12f} &\approx \sqrt{\frac{-m_{1f}}{m_{2f}}} , \\ \theta_{13f} &\approx \frac{m_{2f}}{m_{3f}}\theta_{12f}\theta_{23f} . \end{aligned} \quad (11)$$

This leads to

$$\begin{aligned} V_{cb} &\approx \theta_{23d}-\theta_{23u} , \\ V_{ub} &\approx \theta_{13d}-\theta_{13u}+\theta_{12u}(\theta_{23d}-\theta_{23u}) \sim \theta_{12u}V_{cb} \end{aligned} \quad (12)$$

and (7) for V_{us}. Keeping a grand unified picture in mind, we assume that the M_f in (2) is defined at $M_{\mathrm{GUT}}\sim 10^{16}$ GeV and require it to reproduce the parameters in the quark sector at that scale. For definiteness, we choose the MSSM and quark masses corresponding to $\tan\beta=10$ given in [67].

It follows from (12) that a few percent breaking of the 2–3 symmetry can reproduce the observed mixing quite well for several choices of parameters in M_f. For illustration, we give one specific choice, which is a typical phenomenologically consistent example:

$$\epsilon_{2u} = -\epsilon_{2d} \sim 0.045\,,$$

$$M_d = \begin{pmatrix} -0.003 & 0.0054 & 0.0054 \\ 0.0054 & 0.49 & -0.54 \\ 0.0054 & -0.54 & 0.54 \end{pmatrix},$$

$$M_u = \begin{pmatrix} 0 & 0.0084 & 0.0084 \\ 0.0084 & 42.74 & -41.06 \\ 0.0084 & -41.06 & 39.055 \end{pmatrix}. \qquad (13)$$

These mass matrices lead to the mixing angles $|V_{us}| \approx 0.221$, $|V_{cb}| \approx 0.044$ and $|V_{ub}| \approx 0.0026$. These values are in approximate agreement with the high scale estimates $|V_{us}| \sim 0.223$–0.226, $|V_{cb}| \sim 0.029$–0.038 and $V_{ub} \sim 0.0024$–0.0038 as given for example by Matsuda and Nishiura [30–34]. This agreement can be improved by switching on a small ϵ_1. The approximate 2–3 symmetry of the quark mass matrices is apparent in (13).

3 23 symmetry and leptonic mixing

As argued above, the exact μ–τ symmetry leads to vanishing θ_{23}. This situation can change in the presence of even a small symmetry breaking. In this section we identify two scenarios in which a small μ–τ breaking in the Lagrangian leads to the appearance of a large, almost maximal atmospheric mixing angle. In the first scenario M_l and the effective neutrino mass matrix $\mathcal{M}_\nu$ are simultaneously μ–τ symmetric. This can happen if $\mathcal{M}_\nu$ originates through an approximate 23 symmetric coupling with a Higgs triplet. This assumption can lead to large neutrino mixing provided neutrinos are quasi-degenerate. If the type-I seesaw is operative, than it is more natural to impose (approximate) 23 symmetry on m_{D} and M_{R} rather than on $\mathcal{M}_\nu$ as they originate from the basic couplings in the Lagrangian. $\mathcal{M}_\nu$ is a derived quantity, which needs not even be approximately 23 symmetric even when m_{D} and M_{R} show this symmetry approximately. In this case, one can get a phenomenologically consistent picture with the hierarchical neutrino masses. We discuss these two cases in turn.

3.1 Quasi-degenerate neutrinos

Let us start with the general form (2) for M_l and $\mathcal{M}_\nu$ and assume that the $A_{\nu,l}$ are small parameters as in the case of quarks. Concentrate first on the lower 2×2 block of (2). Its diagonalization gives

$$\epsilon_{2f} = \left(\frac{m_{2f} - m_{3f}}{m_{2f} + m_{3f}}\right) \cos 2\tilde{\theta}_{23f}\,. \qquad (14)$$

$f = l, \nu$ above and $\tan 2\tilde{\theta}_{23f} \equiv \frac{C_f}{\epsilon_{2f} B_f}$ correspond to the 23 mixing angle for f. This equation gives a clue to obtaining approximate 23 symmetry simultaneously for M_l and M_ν and avoiding the vanishing of the atmospheric mixing angle. The approximate 2–3 symmetry requires $\epsilon_{2\nu}, \epsilon_{2l} \ll 1$. For the charged leptons, a small ϵ_{2l} necessarily means $\tilde{\theta}_{23l} \sim \frac{\pi}{4}$ in (14), since m_μ substantially differs from m_τ. In contrast, for neutrinos a small $\epsilon_{2\nu}$ can be realized either with a large $\tilde{\theta}_{23\nu}$ or with $m_{2\nu} \sim m_{3\nu}$. The latter case will correspond to a large atmospheric mixing angle. It follows that in the case of the quasi-degeneracy, there exist ranges in the parameters corresponding to approximately 23 symmetric M_l and $\mathcal{M}_\nu$ and large atmospheric mixing arising due to a small $\tilde{\theta}_{23\nu}$ and almost maximal $\tilde{\theta}_{23l}$. All three neutrinos are required to be quasi-degenerate in order to obtain a simultaneous explanation of the solar and atmospheric neutrino scales. In particular, m_{ν_2} and m_{ν_3} would need to have the same sign to make $\epsilon_{2\nu}$ small.

The 2–3 symmetry can be exact in M_l while it needs to be broken by $\mathcal{M}_\nu$. The amount of the required breaking is quantified using (14):

$$|\epsilon_{2\nu}| \approx \left|\frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_2} + m_{\nu_3}}\right| \approx \left|\frac{\Delta_A}{4m_0^2}\right| \sim 0.08\,, \qquad (15)$$

for the atmospheric scale

$$\Delta_A \sim 3\times 10^{-3}\,\mathrm{eV}^2$$

and the quasi-degenerate mass $m_0 \sim 0.1$ eV. This value is not very different from the symmetry breaking that was required in the quark sector.

In order to analyze the leptonic mixing in the full 3×3 case, let us assume that M_l is 2–3 symmetric and go to the basis with a diagonal M_l. In this basis, the neutrino mass matrix assumes the form

$$\mathcal{M}_{\nu f} \equiv R_{12}^{\mathrm{T}}(\theta_{12l}) R_{23}^{\mathrm{T}}(\pi/4) \mathcal{M}_\nu R_{23}(\pi/4) R_{12}(\theta_{12l})\,. \qquad (16)$$

The θ_{12l} denotes the e–μ mixing, which, in analogy with the quark case, will be assumed to be small, $\theta_{12l} \sim \sqrt{\frac{m_e}{m_\mu}}$. Neglecting its effect, $\mathcal{M}_{\nu f}$ is approximately given by

$$\mathcal{M}_{\nu f} \approx \begin{pmatrix} X_\nu & \sqrt{2}A_\nu & 0 \\ \sqrt{2}A_\nu & B_\nu + C_\nu & \epsilon_{2\nu} B_\nu \\ 0 & \epsilon_{2\nu} B_\nu & B_\nu - C_\nu \end{pmatrix}. \qquad (17)$$

$\mathcal{M}_{\nu f}$ is diagonalized by the PMNS matrix [68, 69] U as follows:

$$U^{\mathrm{T}} \mathcal{M}_{\nu f} U = \mathrm{Diag.}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3})\,, \qquad (18)$$

with $U = R_{23}(\theta_{23}) R_{13}(\theta_{13}) R_{12}(\theta_{12})$ in the standard parameterization.

Consider the symmetric limit corresponding to $\epsilon_{2\nu} = 0$. The quasi-degeneracy $m_{\nu_2} \sim m_{\nu_3}$ is obtained for

$$B_\nu \sim m_0\,, \quad C_\nu \sim \mathcal{O}\left(\frac{\Delta_A}{4m_0}\right). \qquad (19)$$

The atmospheric mixing is zero in this case, but when $\epsilon_{2\nu}$ is turned on, even a small value as given in (15) can lead to a large atmospheric mixing due to the smallness of C_ν. The smallness of C_ν, i.e. the quasi-degeneracy, does not follow from the underlying 2–3 symmetry, but it is quite consistent with it.

The expression for the atmospheric mixing angle follows from the diagonalization of the 23 block

$$\tan 2\theta_{23} = \frac{\epsilon_{2\nu} B_\nu}{C_\nu}\,. \tag{20}$$

This gets a small correction when $A_\nu \sim \mathcal{O}(\frac{\Delta_\odot}{4m_0})$ is turned on.

While the small 2–3 breaking leads to a large atmospheric mixing, U_{e3} remains small. This follows because of the zero in (17) at the (13) entry. Using

$$(\mathcal{M}_{\nu f})_{13} = (UD_\nu U^{\mathrm{T}})_{13} = 0$$

and the quasi-degeneracy, one finds

$$U_{e3} \sim \tan\theta_{23} \sin 2\theta_{12} \frac{\Delta_\odot}{2\Delta_A} \sim \pm 0.03\,, \tag{21}$$

where $\Delta_A \equiv m^2_{\nu_3} - m^2_{\nu_1}$ and $\Delta_\odot \equiv m^2_{\nu_2} - m^2_{\nu_1}$. Note that the normal and inverted neutrino mass hierarchies correspond to opposite signs for U_{e3}.

The above value for U_{e3} would get corrected by (a) the 12 mixing angle in the charged lepton sector, and (b) the symmetry breaking parameter $\epsilon_{1\nu}$, which was also neglected here. The correction due to the angle in (a) gives a contribution [70] of

$$\mathcal{O}\left(\frac{1}{\sqrt{2}}\theta_{12l}\right) \sim 0.05\,,$$

which can add or subtract to the value ~ 0.03 given above depending upon the neutrino mass hierarchy. There may be a relative phase between these contribution in the presence of CP violation. As a consequence, one expects U_{e3} in the present scheme to be typically 0.02–0.08 if $\theta_{12l} \sim \mathcal{O}\left(\frac{1}{\sqrt{2}}\theta_{12l}\right)$. The $\epsilon_{1\nu}$ gives a very small $\sim \mathcal{O}(\frac{\Delta_\odot}{\Delta_A}\epsilon_{1\nu})$ contribution to U_{e3} when $A_\nu \sim \mathcal{O}(\frac{\Delta_\odot}{m_0})$.

The quasi-degeneracy is an essential ingredient in this approach. One would therefore expect a relatively large value for the effective neutrino mass m_{ee} probed by the neutrinoless double beta decay experiments. This is quantified in Fig. 1. The parameters in M_{ν_f} are determined in terms of the lightest neutrino mass m_0, the solar and the atmospheric scales and the corresponding mixing angles using (17) and (18) after imposing (21). These are then varied randomly in their allowed 2σ ranges [71–74] to generate the values of m_{ee} and $\epsilon_{2\nu}$. The sum of the neutrino masses is assumed to be $\leq 0.9\,\mathrm{eV}$ as required by cosmology. One clearly sees that quite large values for m_{ee} are possible, which can be understood from the fact that the scenario corresponds to quasi-degeneracy with all the neutrinos having the same CP property. $\epsilon_{2\nu}$ is restricted in the range ~ 0.005–0.2 with higher m_0 requiring smaller 23 breaking.

The atmospheric mixing angle can be large but it is not required to be maximal as would be the case if only $\mathcal{M}_{\nu f}$

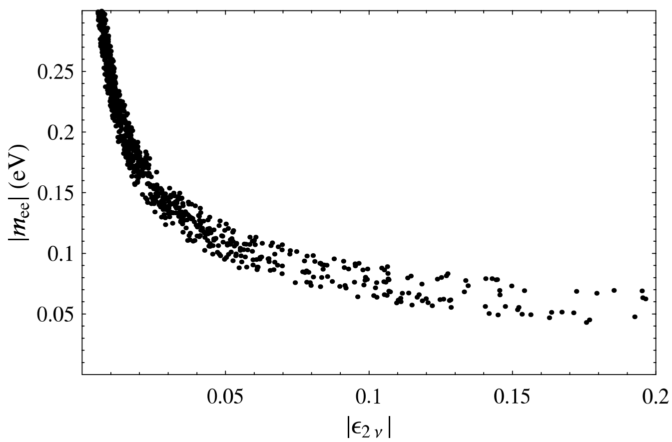

Fig. 1. Allowed ranges of the 23 breaking parameter $\epsilon_{2\nu}$ and the neutrinoless double beta decay mass m_{ee} obtained from (16) with quasi-degenerate spectrum. The solar and the atmospheric scales and mixing angles are randomly varied within their allowed 2σ ranges

were assumed to be μ–τ symmetric. While strict maximality does not obtain, all values allowed by the present data are possible, including values close to the maximal mixing.

3.2 23 symmetry and type-I seesaw

Let us now consider the type-I seesaw model in the presence of approximate 23 symmetry. If both m_{D} and M_{R} are exactly 23 symmetric, then $\mathcal{M}_\nu$ will also be so. But the introduction of even a small 23 breaking in m_{D} and M_{R} can result in a large breaking in $\mathcal{M}_\nu$ with the result that one can get a consistent picture without having a quasi-degenerate spectrum as before. Let us illustrate this with a two generation (μ and τ) example, which can then be generalized easily to the realistic case. We assume that the Lagrangian is approximately 23 symmetric and parameterize the resulting M_l, M_{R} and m_{D} as

$$\begin{aligned} M_{l,\mathrm{D}} &= \frac{m_{l,\mathrm{D}}}{2}\begin{pmatrix} 1-\epsilon_{2l,2\mathrm{D}} & 1+\lambda_{l,\mathrm{D}} \\ 1+\lambda_{l,\mathrm{D}} & 1+\epsilon_{2l,2\mathrm{D}} \end{pmatrix}, \\ M_{\mathrm{R}} &= \frac{m_{\mathrm{R}}}{2}\begin{pmatrix} 1 & 1+\lambda_{\mathrm{R}} \\ 1+\lambda_{\mathrm{R}} & 1 \end{pmatrix}. \end{aligned} \tag{22}$$

$m_{l,\mathrm{D}}$ and m_{R} determine the third generation masses. As before, we have made the additional assumption of symmetry on M_l and m_{D}. The parameters $\epsilon_{2l,2\mathrm{D}}$ signify a small μ–τ breaking. The $\lambda_{l,\mathrm{D}}$ measure the hierarchy in the fermion masses when the terms of $\mathcal{O}(\epsilon_2^2)$ are neglected. Both λ and ϵ_2 are assumed to be much less than one, but they may be comparable to each other. M_{R} is assumed to be μ–τ symmetric in this illustrative example. The effective neutrino mass matrix M_ν in the symmetry basis is given by the seesaw mechanism by

$$\mathcal{M}_\nu = -\frac{m_{\mathrm{D}}^2}{2m_{\mathrm{R}}\lambda_{\mathrm{R}}(2+\lambda_{\mathrm{R}})}\begin{pmatrix} B_\nu(1-\epsilon_{2\nu}) & C_\nu \\ C_\nu & B_\nu(1+\epsilon_{2\nu}) \end{pmatrix}, \tag{23}$$

with

$$\begin{aligned} C_\nu &= (\epsilon_{2\mathrm{D}}^2 - \lambda_\mathrm{D}^2)(1+\lambda_\mathrm{R}) - 2\lambda_\mathrm{R} - 2\lambda_\mathrm{D}\lambda_\mathrm{R}\,, \\ B_\nu &= \epsilon_{2\mathrm{D}}^2 + \lambda_\mathrm{D}^2 - 2\lambda_\mathrm{R} - 2\lambda_\mathrm{R}\lambda_\mathrm{D}\,, \\ \epsilon_{2\nu} &= -\frac{2\epsilon_{2\mathrm{D}}(\lambda_\mathrm{D}+\lambda_\mathrm{R}+\lambda_\mathrm{D}\lambda_\mathrm{R})}{\epsilon_{2\mathrm{D}}^2+\lambda_\mathrm{D}^2-2\lambda_\mathrm{R}-2\lambda_\mathrm{R}\lambda_\mathrm{D}}\,. \end{aligned} \tag{24}$$

The expression of $\epsilon_{2\nu}$ given above shows that $\epsilon_{2\nu}$ can be $\mathcal{O}(1)$ or more for a large ranges in the small parameters $\lambda_{D,R}$ and $\epsilon_{2\mathrm{D}}$, implying that a small 23 breaking in m_D may result in a large symmetry breaking in $\mathcal{M}_\nu$. Consider the hierarchy

$$|\lambda_\mathrm{R}| \ll |\epsilon_{2\mathrm{D}}| \sim |\lambda_\mathrm{D}| \sim \frac{m_{2\mathrm{D}}}{m_{3\mathrm{D}}}\,. \tag{25}$$

This implies $|\epsilon_{2\nu}| \sim 1$ and hence a very large breaking of the μ–τ symmetry in the effective neutrino mass matrix $\mathcal{M}_\nu$. There are two important consequences of this large breaking. Firstly, the neutrino mass hierarchy is normal when (25) is obeyed:

$$\frac{m_{\nu_2}}{m_{\nu_3}} \approx -\frac{\lambda_\mathrm{R}(\epsilon_{2\mathrm{D}}^2+\lambda_\mathrm{D}^2+2\lambda_\mathrm{D})^2}{2(\epsilon_{2\mathrm{D}}^2+\lambda_\mathrm{D}^2)^2}\,. \tag{26}$$

Secondly, the mixing in the neutrino mass matrix is also suppressed:

$$\tan 2\theta_{23\nu} \approx \frac{(\epsilon_{2\mathrm{D}}^2-\lambda_\mathrm{D}^2)(1+\lambda_\mathrm{R})-2\lambda_\mathrm{R}-2\lambda_\mathrm{R}\lambda_\mathrm{D}}{2\epsilon_{2\mathrm{D}}(\lambda_\mathrm{D}+\lambda_\mathrm{R}+\lambda_\mathrm{D}\lambda_\mathrm{R})}\,. \tag{27}$$

The mixing in the charged lepton mass matrix (22) is, however, large when $\epsilon_{2l} \ll 1$. As a consequence, the atmospheric mixing angle remains large.

In spite of the apparent large μ–τ breaking in M_ν, the μ–τ violation remains small as far as the physical observables are concerned. This is best seen by going to the flavor basis. Assuming the exact μ–τ symmetry in M_l, i.e. $\epsilon_{2l}=0$ and hence the maximal θ_{23l}, the neutrino mass matrix in the flavor basis assumes the following form:

$$\mathcal{M}_{\nu f} = -\frac{m_\mathrm{D}^2}{2m_\mathrm{R}\lambda_\mathrm{R}(2+\lambda_\mathrm{R})} \times \begin{pmatrix} B_{\nu f}(1-\epsilon_{2\nu f}) & C_{\nu f} \\ C_{\nu f} & B_{\nu f}(1+\epsilon_{2\nu f}) \end{pmatrix}, \tag{28}$$

with

$$\begin{aligned} C_{\nu f} &= -2\epsilon_{2\mathrm{D}}(\lambda_\mathrm{D}+\lambda_\mathrm{R}+\lambda_\mathrm{R}\lambda_\mathrm{D})\,, \\ B_{\nu f} &= B_\nu\,, \\ \epsilon_{2\nu f} &= \frac{(\lambda_\mathrm{D}^2-\epsilon_{2\mathrm{D}}^2)(1+\lambda_\mathrm{R})+2\lambda_\mathrm{R}+2\lambda_\mathrm{D}\lambda_\mathrm{R}}{\epsilon_{2\mathrm{D}}^2+\lambda_\mathrm{D}^2-2\lambda_\mathrm{R}-2\lambda_\mathrm{D}\lambda_\mathrm{R}}\,. \end{aligned} \tag{29}$$

The μ–τ breaking in this mass matrix is now characterized by $\epsilon_{2\nu f}$, which, unlike $\epsilon_{2\nu}$, is a small parameter when the couplings obey the hierarchy as in (25). This smallness implies large atmospheric mixing. The seesaw mechanism has thus produced a nearly μ–τ symmetric effective $\mathcal{M}_{\nu f}$ in the flavor basis with diagonal charged lepton masses. The masses of μ and τ are not related by the underlying symmetry and do not in anyway imply a large μ–τ breaking at the fundamental level. In fact, all terms in the basic Lagrangian are approximately μ–τ symmetric with breaking specified by the only parameter $|\epsilon_{2\mathrm{D}}| \ll 1$ in the present case.

The mechanism implemented here in the presence of small μ–τ breaking is a special case of the general analysis of the seesaw enhancement of the leptonic mixing considered in [75–77]. The seesaw enhancement (in angle θ_{ss} in the notation of [75–77]) occurring due to the singular nature of M_R and the hierarchy in the Dirac masses (almost) cancels here with the large left handed mixing in the Dirac sector, with the result that the neutrino mass matrix $\mathcal{M}_\nu$ in the μ–τ symmetric basis is characterized by a small mixing. The maximal mixing through M_l due to μ–τ symmetry then accounts for the atmospheric mixing angle.

The above example is fairly realistic. Its generalization to three generations must be such that θ_{13} remains small. In this case, the atmospheric mixing angle can be approximately given by (28):

$$\tan 2\theta_{23} \approx \frac{2\epsilon_{2\mathrm{D}}(\lambda_\mathrm{D}+\lambda_\mathrm{R}+\lambda_\mathrm{D}\lambda_\mathrm{R})}{(\lambda_\mathrm{D}^2-\epsilon_{2\mathrm{D}}^2)(1+\lambda_\mathrm{R})+2\lambda_\mathrm{R}+2\lambda_\mathrm{D}\lambda_\mathrm{R}}\,, \tag{30}$$

which is large when (25) is obeyed. If the third neutrino mass remains smaller than the two masses here, then the ratio r_Δ of the solar to atmospheric mass is given by (26). θ_{23} and this mass ratio together provide a constraint on the relative values of the hierarchy parameters in (25). Specifically, define $x=\frac{\epsilon_{2\mathrm{D}}}{\lambda_\mathrm{D}}$ and $y=\frac{\lambda_\mathrm{R}}{\lambda_\mathrm{D}^2}$. There exist several values of the pair x,y that can reproduce r_Δ and $\sin^2 2\theta_{23}$; e.g., choosing $\lambda \sim 0.22$, we find

$$\begin{aligned} &|x| \sim 0.748\,, \quad y \sim -0.197\,, \quad |x| \sim 1.27\,, \quad y \sim 0.291\,, \\ &|x| \sim 0.713\,, \quad y \sim -0.182\,, \quad |x| \sim 1.37\,, \quad y \sim 0.33\,. \end{aligned} \tag{31}$$

These reproduce the best fit values:

$$r_\Delta \approx \left|\frac{m_{\nu_2}}{m_{\nu_3}}\right| \sim 0.18\,; \quad \sin^2 2\theta_{23} \approx 0.999\,.$$

The above solutions x,y depend only mildly on the choice of λ_D. The other solutions correspond to $x,y \gg 1$ and hence large μ–τ breaking ($\epsilon_\mathrm{D} \gg 1$) for hierarchical Dirac masses, i.e. $\lambda_\mathrm{D}, 1 \ll 1$.

The generalization of (22) to three generations can be defined by

$$\begin{aligned} M_{l,\mathrm{D}} &= \frac{m_{l,\mathrm{D}}}{2}\begin{pmatrix} X_{l,\mathrm{D}} & A_{l,\mathrm{D}} & A_{l,\mathrm{D}} \\ A_{l,\mathrm{D}} & 1-\epsilon_{2l,2\mathrm{D}} & 1+\lambda_{l,\mathrm{D}} \\ A_{l,\mathrm{D}} & 1+\lambda_{l,\mathrm{D}} & 1+\epsilon_{2l,2\mathrm{D}} \end{pmatrix}, \\ M_\mathrm{R}^{-1} &= \frac{2}{m_\mathrm{R}}\begin{pmatrix} X_\mathrm{R} & A_\mathrm{R} & A_\mathrm{R} \\ A_\mathrm{R} & 1 & -(1+\lambda_\mathrm{R}) \\ A_\mathrm{R} & -(1+\lambda_\mathrm{R}) & 1 \end{pmatrix}. \end{aligned} \tag{32}$$

We have assumed $M_{l,\mathrm{D}}$ to be symmetric as before. One could have added one more μ–τ breaking parameter in the (12) and (13) elements in M_D, which we have set to zero for simplicity. Likewise M_{R} is also assumed to be μ–τ symmetric. The parameterization of M_{R}^{-1} is chosen above in such a way that this effectively reproduces M_{R} (up to a constant) in the case of two generations when $A_{\mathrm{R}} \ll 1$. In practice, A_{R} and X_{R} need not be small and can even provide more dominant entries in M_{R}^{-1}. We will allow them to be arbitrary.

The effective neutrino mass matrix after the seesaw mechanism can be parameterized as in (23):

$$\mathcal{M}_\nu = -\frac{m_{\mathrm{D}}^2}{2m_{\mathrm{R}}} \begin{pmatrix} X_\nu & A_\nu(1-\epsilon_{1\nu}) & A_\nu(1+\epsilon_{1\nu}) \\ A_\nu(1-\epsilon_{1\nu}) & B_\nu(1-\epsilon_{2\nu}) & C_\nu \\ A_\nu(1+\epsilon_{1\nu}) & C_\nu & B_\nu(1+\epsilon_{2\nu}) \end{pmatrix} . \tag{33}$$

The parameters defined above can be related to the ones in (32) in a straightforward manner. A particularly interesting limit corresponds to neglecting of A_{D} in M_{D}. If the neutrino Dirac masses are to be hierarchical, then this would be a good limit and we give our analytic results assuming $A_{\mathrm{D}} = 0$. We have, however, retained A_{D} in the numerical results to be presented. The essential features of the two generation example are preserved in the limit of A_{D} going to zero, but one still has enough freedom to generate the solar mixing angle and the third neutrino mass. In particular, B_ν, C_ν and $\epsilon_{2\nu}$ are given by the same expression as in (24). The other new parameters entering (33) are given by

$$\begin{aligned} X_\nu &= X_{\mathrm{D}}^2 X_{\mathrm{R}} , \\ A_\nu &= X_{\mathrm{D}} A_{\mathrm{R}} (2+\lambda_{\mathrm{D}}) , \\ \epsilon_{1\nu} &= \frac{\epsilon_{\mathrm{D}}}{2+\lambda_{\mathrm{D}}} . \end{aligned} \tag{34}$$

As before we neglect μ–τ breaking in the charged lepton masses and neglect θ_{12l}. In this limit, one can go to the flavor basis through a rotation by $\pi/4$ in the 23 plane:

$$\mathcal{M}_{\nu f} = R_{23}^{\mathrm{T}}(\pi/4) M_\nu R_{23} , \tag{35}$$

where M_ν is the 3×3 matrix given in (33).

Diagonalization of $\mathcal{M}_{\nu f}$ directly gives the MNS matrix. This can be achieved through three successive rotations:

$$U^{\mathrm{T}} M_\nu U = \mathrm{diag.}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) , \tag{36}$$

where

$$U \equiv R_{23}(\theta_{23}) R_{13}(\theta_{13}) R_{12}(\theta_{12}) .$$

The neutrino masses are approximately given by

$$\begin{aligned} m_{\nu_{1,2}} &\approx \frac{1}{2}\left(X_\nu + \lambda_2 \pm \frac{X_\nu - \lambda_2}{\cos 2\theta_{12}} \right) , \\ m_{\nu_3} &\approx B_\nu - \frac{C_\nu}{\cos 2\theta_{23}} . \end{aligned} \tag{37}$$

The mixing angle θ_{23} remains approximately the same as in (30). The others are given by

$$\begin{aligned} \tan 2\theta_{13} &\approx \frac{2\sqrt{2} A_\nu (c_{23}\epsilon_{1\nu} - s_{23})}{X_\nu - m_{\nu_3}} , \\ \tan 2\theta_{12} &\approx \frac{2\sqrt{2} A_\nu (c_{23} + s_{23} e_{1\nu})}{X_\nu - \lambda_2} , \end{aligned} \tag{38}$$

where

$$\lambda_2 = B_\nu + \frac{C_\nu}{\cos 2\theta_{23}} . \tag{39}$$

We have neglected contributions of $\mathcal{O}(s_{13}^2)$ in writing the above equations.

These equations provide a very good approximation to the actual masses and mixing. From (30) and (38) we have the following.

- The atmospheric mixing angle is large when the hierarchy (25) is satisfied as in the case of the two generations.
- The hierarchy in the neutrino masses m_{ν_1,ν_2} need not be strong but the m_{ν_3} is still the dominant mass because of (25) and the ratio of solar to atmospheric mass can be reproduced as in the two generation case, but now with a wider parameter space.
- The hierarchy (25) implies $|\lambda_2| \ll |m_{\nu_3}|$. As a consequence, one can simultaneously get a suppressed θ_{13} and a large θ_{12} if $|X_\nu| \ll |\lambda_2| \sim |2A_{2\nu}|$.

With the introduction of the third generation, the allowed patterns of the right handed neutrino mass matrix have a considerable variation compared to the case of two generations. We have explored this numerically. We have varied all parameters in M_{D} and M_{R} randomly and obtained acceptable solutions requiring that (i) the solar and the atmospheric mixing angles and (mass)^2 differences remain within the 2σ range and $\theta_{13} < 0.17$; (ii) the masses of M_{D} follow the hierarchical pattern, and (iii) μ–τ symmetry breaking is mild, $|\epsilon_{2\mathrm{D}}| \leq 0.2$. The charge lepton mass matrix was assumed μ–τ symmetric and θ_{12l} was neglected. From the numerical analysis, we find the following.

- The neutrino contribution to θ_{13} remains relatively large, ≥ 0.07; see specific examples in the appendix.
- $\cos 2\theta_{23}$ can take all values within the allowed range 0–0.3. Strictly maximal mixing is not possible but values very close to it occur frequently.
- Many examples correspond to the very small symmetry breaking $|\epsilon_{2\nu}| \leq$ few %.
- Unlike the two generation example, the RH neutrino masses are not required to show strong hierarchy. Different mass patterns – hierarchical, pseudo-Dirac pair or completely quasi-degenerate spectrum – are possible. We give an example of these in the appendix.

4 Realization

We now turn to a concrete realization of our basic ansatz M_f given in (2). This can be derived in a straightforward manner within the standard two double model by imposing a 2–3 symmetry on the Yukawa couplings. One of the doublets (ϕ_1) is assumed to be invariant, while the other one (ϕ_2) is odd under the 2–3 symmetry. The Yukawa couplings for a fermion f are then given by

$$-\mathcal{L}_Y = \bar{f}_L (\Gamma_1 \phi_1 + \Gamma_2 \phi_2) f_{\mathrm{R}} + \mathrm{H.c.} \tag{40}$$

The (assumed) symmetry of $\Gamma_{1,2}$ and the 2–3 symmetry together lead to the matrix M_f. The Γ_1 generates the parameters A_f, B_f and C_f in (2), and Γ_2 generates $\epsilon_{1f,2f}$ terms. The smallness of $\epsilon_{1f,2f}$ compared to the leading elements can be obtained by assuming the corresponding elements of $\Gamma_{1,2}$ to be similar but taking $\frac{\langle\phi_2\rangle}{\langle\phi_1\rangle}$ to be small, ≤ 0.1.

The above Yukawa terms generate all the approximately 23 symmetric Dirac mass matrices. The $M_{\rm R}$ can come from a 23 symmetric explicit mass term in the standard model or from a coupling of fermions to a 126 field transforming trivially under the 23 symmetry. This together with the approximate 23 symmetric $m_{\rm D}$ realizes scenario II. Realization of the first scenario requires a dominant type-II contribution to the neutrino masses and a quasi-degenerate neutrino spectrum.

5 Summary

In summary, let us recapitulate the salient features of the scheme.

- We showed that the μ–τ symmetry can be extended to all fermions with interesting consequences. Its imposition in the quark sector provides an explanation of the smallness of V_{cb} and V_{ub} compared to the Cabibbo angle. The latter can naturally be explained if an additional symmetry D as defined in (6) is imposed. This needs to be broken badly by the effective neutrino mass matrix in order to get the quasi-degenerate spectrum.
- Our approach to the realization of the μ–τ symmetry in the leptonic sector is quite different from the one considered in the literature. Almost all the previous works do not start with μ–τ symmetry at the fundamental level. Either one just assumes that this symmetry is present in the effective neutrino mass matrix in the flavor basis or starts with a bigger symmetry (e.g. D_4 and S_3), which leads to such an effective mass matrix. We have shown that one can obtain such an effective mass matrix from the approximate 23 symmetry at the fundamental level. Breaking of the 23 symmetry is crucial in the absence of which one gets a zero atmospheric angle. But as discussed here, one does not need to introduce a very large breaking to obtain almost maximal mixing and a few % breaking is sufficient to obtain the correct mixing pattern. We identified two scenarios consistent with mildly broken 23 symmetry. The first one works only if the neutrino spectrum is quasi-degenerate. The other one follows from the conventional seesaw mechanism. We have numerically identified various patterns of $M_{\rm R}$ and $m_{\rm D}$ (see the appendix) that lead to almost maximal atmospheric mixing in spite of very mild breaking of the 23 symmetry.
- The first scenario generally predicts a large amplitude in neutrinoless double beta decay, a small U_{e3} and almost maximal θ_{23}. The second scenario based on the conventional seesaw predicts a measurable U_{e3} and a hierarchical neutrino spectrum.
- The μ–τ symmetry has been extended to the quark sector in some of the earlier works also [30–34]. These relied on breaking of the μ–τ symmetry through a phase in the mass matrix. Thus CP violation was essential to obtain phenomenological consistency. In the present case one can obtain a correct picture even if CP is exact.
- The Yukawa couplings in (40) generate the flavor changing neutral currents (FCNC). One finds that the specific structures of the Yukawa couplings $\Gamma_{1,2}$ lead to the hierarchical strengths ($|F_{12}| \ll |F_{13}| \ll |F_{23}|$) for the FCNC current couplings F_{ij} between flavors i and j to the Higgs in the case with one parameter symmetry breaking, i.e., with $\epsilon_2 \neq 0$. Rough estimates give for the down quarks $F_{12} \sim \frac{m_b}{v}\lambda^5$, $F_{13} \sim \frac{m_b}{v}\lambda^3$ and $F_{23} \sim \frac{m_b}{v}\lambda^2$, where $\lambda \sim 0.22$ and v is the weak scale. Because of this suppression, the FCNC do not require an unusually large Higgs mass. A similar hierarchy in FCNC is found in some Higgs doublets models [78–80] and in Z couplings [81] in models with additional quarks. Consequences of this FCNC are discussed in [82].
- The entire scenario is compatible with grand unification and can be embedded in theories such as SO(10).

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Appendix

We have identified various approximately μ–τ symmetric patterns of $m_{\rm D}$ and $M_{\rm R}$ that lead to a large atmospheric mixing angle. The Dirac neutrino masses are always hierarchical in these examples. One can identify basically three different patterns for $M_{\rm R}$ corresponding to different types of spectrum for $M_{\rm R}$. Examples of these are given in this appendix.

Appendix A

Many of the textures retain their approximate two generation forms and have a dominant block in the 2–3 sector. This is exemplified by the following textures:

$$M_{\rm R} = \frac{m_{\rm R}}{2}\begin{pmatrix} -0.002 & 1.25 & 1.25 \\ 1.25 & -30.43 & -30.92 \\ 1.25 & -30.92 & -30.43 \end{pmatrix},$$
$$M_{\rm D} = \frac{m_{\rm D}}{2}\begin{pmatrix} 0.0042 & 0.00064 & 0.00064 \\ 0.00064 & 0.93 & 1.10 \\ 0.00064 & 1.10 & 1.07 \end{pmatrix}. \quad \text{(A.1)}$$

The resulting light neutrino mass matrix is

$$\mathcal{M}_\nu = \begin{pmatrix} 0.00065 & 0.0063 & 0.0068 \\ 0.0063 & 0.049 & -0.00043 \\ 0.0068 & -0.00043 & 0.0052 \end{pmatrix}, \quad \text{(A.2)}$$

giving

$$\begin{aligned} &\epsilon_{2\mathrm{D}} \sim 0.07\,, \\ &\tan^2\theta_{12} \approx 0.46\,; \quad \sin^2 2\theta_{23} \approx 0.999\,, \\ &\sin\theta_{13} \approx 0.13\,; \quad \Delta_\odot \sim 7.4\times 10^{-5}\,\mathrm{eV}^2\,. \end{aligned} \tag{A.3}$$

$\frac{m_{\mathrm{D}}^2}{m_{\mathrm{R}}}$ is normalized in the above equations to obtain the atmospheric mass scale $\Delta_A = 0.0025\,\mathrm{eV}^2$.

Appendix B

Several textures display an approximate L_e–L_μ–L_τ symmetry in the right handed sector. This is exemplified by

$$\begin{aligned} M_{\mathrm{R}} &= \frac{m_{\mathrm{R}}}{2}\begin{pmatrix} 0.06 & 6.52 & 6.52 \\ 6.52 & 0.10 & -0.40 \\ 6.52 & -0.40 & 0.10 \end{pmatrix}\,, \\ M_{\mathrm{D}} &= \frac{m_{\mathrm{D}}}{2}\begin{pmatrix} 0.0072 & 0.00003 & 0.00003 \\ 0.00003 & 1.06 & 0.96 \\ 0.00003 & 0.96 & 0.94 \end{pmatrix}\,. \end{aligned} \tag{B.1}$$

The resulting light neutrino mass matrix is

$$\mathcal{M}_\nu = \begin{pmatrix} 1.4\times 10^{-6} & 0.0066 & 0.0062 \\ 0.0066 & 0.049 & 0.0007 \\ 0.0062 & 0.0007 & -0.0061 \end{pmatrix}\,, \tag{B.2}$$

giving

$$\begin{aligned} &\epsilon_{2\mathrm{D}} \sim -0.06\,, \\ &\tan^2\theta_{12} \approx 0.44\,, \quad \sin^2 2\theta_{23} \approx 0.997\,, \\ &\sin\theta_{13} \approx 0.13\,, \quad \Delta_\odot \sim 9.2\times 10^{-5}\,\mathrm{eV}^2\,. \end{aligned} \tag{B.3}$$

Appendix C

The third category consists of the non-hierarchical right handed neutrino masses:

$$\begin{aligned} M_{\mathrm{R}} &= \frac{m_{\mathrm{R}}}{2}\begin{pmatrix} -0.00024 & 0.21 & 0.21 \\ 0.21 & 0.22 & -0.28 \\ 0.21 & -0.28 & 0.22 \end{pmatrix}\,, \\ M_{\mathrm{D}} &= \frac{m_{\mathrm{D}}}{2}\begin{pmatrix} 0.002 & 5.7\times 10^{-6} & 5.7\times 10^{-6} \\ 5.7\times 10^{-6} & 1.085 & 0.87 \\ 5.7\times 10^{-6} & 0.87 & 0.915 \end{pmatrix}\,. \end{aligned} \tag{C.1}$$

The resulting light neutrino mass matrix is

$$\mathcal{M}_\nu = \begin{pmatrix} 2.6\times 10^{-6} & 0.0091 & 0.0083 \\ 0.0091 & 0.049 & -0.004 \\ 0.0083 & -0.004 & 0.006 \end{pmatrix}\,, \tag{C.2}$$

giving

$$\begin{aligned} &\epsilon_{2\mathrm{D}} \sim -0.085\,, \\ &\tan^2\theta_{12} \approx 0.46\,, \quad \sin^2 2\theta_{23} \approx 0.988\,, \\ &\sin\theta_{13} \approx 0.17\,, \quad \Delta_\odot \sim 8.0\times 10^{-5}\,\mathrm{eV}^2\,. \end{aligned} \tag{C.3}$$

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